

Quantum catastrophes II. Generic pattern of the fall into instability

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Abstract

The quantum-catastrophe (QC) benchmark Hamiltonians of paper I (M. Znojil, J. Phys. A: Math. Theor. 45 (2012) 444036) are reconsidered, with the infinitesimal QC distance λ replaced by the total time $\tau = \sqrt{1 - \lambda} \in (0, 1)$ of the fall into the singularity. Our amended model becomes unique, describing the complete QC history as initiated by a Hermitian and diagonalized N -level oscillator Hamiltonian at $\tau = 0$. In the limit $\tau \rightarrow 1$ the system finally collapses into the completely (i.e., N -times) degenerate QC state. The closed and compact Hilbert-space metrics $\Theta^{(N)}(\tau)$ are then calculated and displayed up to $N = 7$. The phenomenon of the QC collapse is finally attributed to the manifest time-dependence of the Hilbert space and, in particular, to the emergence and to the growth of its anisotropy. A quantitative measure of such a time-dependent anisotropy is found in the spread of the N -plet of the eigenvalues $\theta_n^{(N)}(\tau)$ of the metric. Unexpectedly, the model appears exactly solvable – at any multiplicity N , the N -plet of eigenvalues $\theta_n^{(N)}(\tau)$ is obtained in closed form.

1 Introduction

In paper I [1] we analyzed bound-state energies $E_n^{(N)}(\lambda)$, $n = 0, 1, \dots, N - 1$ generated by certain toy-model Hamiltonians $H^{(N)}(\lambda)$. A characteristic property of our model was that an infinitesimal decrease of the real parameter λ below critical λ_0 implied an abrupt complexification of the whole spectrum. Such a multifurcation quantum catastrophe (QC) was interpreted as an N -tuple analogue of the Thom's bifurcation catastrophe which often occurs in classical dynamical systems and which is usually called "cusp" [2].

The transition from classical to quantum catastrophes proved facilitated, first of all, by the recent new developments in quantum theory. A review of these developments may be found, e.g., in Ref. [3] (where the underlying innovative representations of the operators of observables were introduced and called quasi-Hermitian) or in Ref. [4] (where the author, very successfully, recommended the use of special, \mathcal{PT} -symmetric and \mathcal{PCT} -symmetric quasi-Hermitian Hamiltonians, with \mathcal{P} representing parity, with \mathcal{C} being a charge, and with symbol \mathcal{T} denoting the usual time reversal) or in Refs. [5] (where one can find an immediate connection of the new physics with the traditional mathematical theory of operators which are self-adjoint in Krein space).

In Appendix A of paper I the quasi-Hermiticity property of the Hamiltonians $H^{(N)}(\lambda)$ was re-christened into "crypto-Hermiticity", replacing its widespread but rather misleading nickname of "non-Hermiticity". A compact account of the terminology may be also found in Ref. [6] where we recommended to call the whole formalism the "three-Hilbert-space (THS) quantum mechanics". In our present paper we intend to make the next step in connecting such a THS formalism to the currently open problem of the definition and systematic description of catastrophic scenarios in quantum theory.

Several topics will be addressed. Firstly, in section 2 we shall briefly summarize the preceding developments and recall some properties of our benchmark bounded-operator Hamiltonians. Next, in section 3 we shall discuss the necessary specification of the suitable physical Hilbert space, paying particular attention to the ambiguity of the definition of the mathematically correct inner product in this space. Our present innovative, QC-adapted recipe of the removal of this ambiguity will be thoroughly analyzed in section 4 where we shall present arguments in favor of the choice of parameters which would, in some sense, minimize the anisotropy of the physical Hilbert-space metric Θ . We shall also replace there the time t of paper I (which starts at the QC singularity) by the more natural time τ which

runs in opposite direction and which is assumed to have its initial zero long before the fall of the system into the final and singular QC degeneracy.

In section 5 we then turn attention to the systems with the smallest level-multiplicities $N \leq 4$. Using the brute-force linear algebra methods we shall construct the fully explicit forms of the matrices of the metrics which we declared, in the preceding text, optimal and unique. On this background, the qualitative features of the QC process will be shown related to the properties of our manifestly time-dependent physical Hilbert spaces.

In section 6 the transparency and compact form of the previous results and formulae will enable us to employ extrapolation techniques. Their use will extend the validity of many of our previous considerations to arbitrary N . The whole picture of the N -tuple QC level-degeneracy scenario will be finally summarized in section 7.

2 Benchmark models

2.1 Complexified potentials

In the current literature the applicability of the THS formalism is most often illustrated by the replacement of the linear harmonic oscillator Hamiltonian $H^{(LHO)} = p^2 + x^2$ by the Bender's and Boettcher's [7] set of anharmonic models $H^{(BB)}(\delta) = p^2 + i^\delta x^{2+\delta}$. They are distinguished by the real exponent $\delta \geq 0$. In a historical perspective their study appeared well motivated by the needs of quantum field theory (cf., e.g., [8]) and of various perturbation expansion methods [9, 10].

In contrast to the exceptional selfadjoint operator $H^{(LHO)} = H^{(BB)}(0)$ which lives in the most common representation $L^2(\mathbb{R})$ of the physical Hilbert space of states, the correct physical interpretation of the apparently non-Hermitian Hamiltonian $H^{(BB)}(\delta)$ requires an *ad hoc* specification of the correct physical Hilbert space of states $\mathcal{H}^{(S)} \neq L^2(\mathbb{R})$ [4]. In other words, the correct probabilistic quantum predictions are only obtained when one *redefines* the inner product in the vector space of the square-integrable wave functions in a way explained in Refs. [3, 6],

$$\langle \psi_1 | \psi_2 \rangle \rightarrow \langle \psi_1 | \psi_2 \rangle_{(S)} := \langle \psi_1 | \Theta^{(S)} | \psi_2 \rangle. \quad (1)$$

Needless to add that the metric $\Theta^{(S)} \neq I$ must satisfy severe mathematical restrictions as listed in Refs. [3, 11]. Only then, the standard physical Hilbert space $\mathcal{H}^{(S)}$ becomes well

defined and the Hamiltonian itself may be declared self-adjoint in this space.

The use of nontrivial and, in general, manifestly Hamiltonian-dependent Hilbert-space-metric operators may be perceived as an important innovation of the model-building in quantum theory. Nevertheless, whenever accepted as a sound theoretical tool in physics, its mathematical consistency must always be carefully reexamined. In this sense the recent rigorous mathematical proof [12] of the nonexistence of the metric operator for the most popular and phenomenologically highly relevant $H^{(BB)}(\delta)$ makes the study of this particular Hamiltonian far from being completed. For illustration purposes, fortunately, various other, mathematically simpler and fully consistent models are available and may be found in the current literature [3, 13].

2.2 Bounded-operator models

One of the main and well known [11] difficulties accompanying the use of the Hamiltonians which are defined as differential operators lies in the necessity of a guarantee of the boundedness of the metric operator $\Theta^{(S)}$ in Eq. (1). Many authors are trying to avoid this purely mathematical obstacle by the assumption of the boundedness of the operators of observables [3]. In this spirit we replaced, in paper [14], the above-mentioned ordinary differential Hamiltonian $H^{(LHO)}$ by its truncated and shifted, N -dimensional diagonal-matrix simplification

$$H_{(LHO)}^{(N)} = \begin{bmatrix} -(N-1) & 0 & \dots & 0 \\ 0 & -(N-3) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & +(N-1) \end{bmatrix}. \quad (2)$$

In parallel, the methodical and pedagogical role of the Bender's and Boettcher's Hermiticity-violating differential-operator Hamiltonian $H^{(BB)}(\delta)$ was transferred there to the multi-parametric anharmonic-oscillator-like real and tridiagonal non-Hermitian N by N matrices

$$H_{(AHO)}^{(N)} = \begin{bmatrix} 1-N & g_1 & 0 & 0 & \dots & 0 \\ -g_1 & 3-N & g_2 & 0 & \dots & 0 \\ 0 & -g_2 & 5-N & \ddots & \ddots & \vdots \\ 0 & 0 & -g_3 & \ddots & g_{N-2} & 0 \\ \vdots & \vdots & \ddots & \ddots & N-3 & g_{N-1} \\ 0 & 0 & \dots & 0 & -g_{N-1} & N-1 \end{bmatrix}. \quad (3)$$

This allowed us to avoid the necessity of the complicated proof of the reality of the bound-state spectra of $H^{(BB)}(\delta)$ (cf. [15]). For our models (3) we reduced the proof to a virtually elementary spectral-continuity or spectral-inertia argument, applicable at the not too large couplings g_j at least. Moreover, after an additional up-down symmetrization of the Hamiltonian matrix, i.e., after the choice of $g_{N-1} = g_1$ and $g_{N-2} = g_2$, etc, we arrived at the final form of our benchmark toy-model matrices

$$H_{(PT)}^{(N)} = \begin{bmatrix} 1-N & g_1 & 0 & 0 & \dots & 0 \\ -g_1 & 3-N & g_2 & 0 & \dots & 0 \\ 0 & -g_2 & 5-N & \ddots & \ddots & \vdots \\ 0 & 0 & -g_3 & \ddots & g_2 & 0 \\ \vdots & \vdots & \ddots & \ddots & N-3 & g_1 \\ 0 & 0 & \dots & 0 & -g_1 & N-1 \end{bmatrix}. \quad (4)$$

With such a choice we paralleled the phenomenologically relevant parity times time reversal symmetry of the above-mentioned toy-model Hamiltonians $H^{(BB)}(\delta)$ by a formally analogous \mathcal{PT} -symmetry of our matrices (4). This merely required the specification of \mathcal{T} as the transposition plus sign reversal. The parity-simulating indefinite square root of the unit matrix appeared then represented by the antidiagonal, N by N matrix \mathcal{P} with elements $\mathcal{P}_{j,N-j+1} = 1$, $j = 1, 2, \dots, N$.

In our subsequent paper [16] we turned attention to one of the methodically most welcome features of the \mathcal{PT} -symmetric and $J = [N/2]$ -parametric benchmark Hamiltonian (4), viz., to the amazingly elementary geometric form of the boundary $\partial\mathcal{D}$ of the J -dimensional compact domain \mathcal{D} of the real parameters g_j for which the spectrum of $H_{(PT)}^{(N)}$ remains real. For our models (4) this boundary (or, in the language of physics, the horizon of the bound-state stability of the system [17]) has been shown to acquire, at any matrix dimension N , the same generic geometric form of the surface of a smoothly deformed hypercube with protruded (hyper)edges and vertices (cf. also [18, 19] for more details).

In paper I we finally restricted our attention to models (4) in the strongest-coupling extreme. In a sufficiently small vicinity of any one of the vertices we revealed that there exists a certain specific λ -parametrization of the couplings $g_j(\lambda)$ such that

- the two by two matrix $H_{(PT)}^{(2)}(\lambda)$ appears useful as a benchmark model of an *energy-*

bifurcation QC scenario in which

$$E_0 = -\sqrt{\lambda}, \quad E_1 = +\sqrt{\lambda}, \quad (5)$$

i.e., in which the spectrum is real iff $\lambda \geq \lambda_0 = 0$ and in which the whole spectrum becomes completely degenerate iff $\lambda = 0$ while it finally gets purely imaginary iff $\lambda < 0$;

- the three by three matrix $H_{(PT)}^{(3)}(\lambda)$ may be used as a benchmark model of a new, *energy-trifurcation* quantum catastrophe in which

$$E_0 = -2\sqrt{\lambda}, \quad E_1 = 0, \quad E_2 = +2\sqrt{\lambda}. \quad (6)$$

Again, the spectrum proves completely degenerate iff $\lambda = 0$. Up to the exceptional, λ -independent real level $E_{[N/2]} = 0$ emerging at any odd N , the rest of the spectrum is also purely real or imaginary iff $\lambda \geq 0$ or $\lambda < 0$, respectively;

- the four by four matrix $H_{(PT)}^{(4)}(\lambda)$ may then serve as a benchmark quantum model of the analogous *energy-quadrifurcation* catastrophe in which

$$E_0 = -3\sqrt{\lambda}, \quad E_1 = -\sqrt{\lambda}, \quad E_2 = \sqrt{\lambda}, \quad E_3 = 3\sqrt{\lambda}, \quad (7)$$

etc. Same conclusions were found to hold, for Eq. (4), at any integer $N \geq 2$.

In what follows we intend to develop further the emerging idea of an N -furcating quantum-theory generalization of the classical bifurcation catastrophe. In this direction our present paper will clarify, first of all, some of the problems which emerge during the necessary reconstruction of the physical Hilbert-space metric $\Theta = \Theta(\lambda)$.

3 Quantum theory near the singularity

3.1 The concept of the time of recovery $t > 0$

One of the simplest forms of the above-mentioned λ -parametrizations of the couplings in Hamiltonians (4) is given by the formula which was proposed in paper [16],

$$g_n^{(PT)}(\lambda) = \sqrt{n(N-n)(1-\lambda)} \in \mathcal{D}, \quad n = 1, 2, \dots, N-1. \quad (8)$$

The merit of this parametrization is that in the whole interval of $\lambda \in (0, 1)$ (or formally even of $\lambda \in (0, \infty)$) it guarantees the reality of the energy spectrum by keeping the whole J -plet of couplings safely inside the physical domain \mathcal{D} . The second merit of the λ -parametrization (8) lies in the fact that the boundary value of $\lambda = 0$ strictly separates the dynamical quantum $\lambda > 0$ regime yielding the real, “observable” N -plet of bound state energies from the completely unobservable $\lambda < 0$ half-axis. Geometrically this means that the value of $\lambda = 0$ localizes the protruded QC vertex of the domain \mathcal{D} . In the algebraic terminology of original paper [14] and of the older literature [20, 21] one encounters the so called extreme exceptional point (EEP) at $\lambda = 0$.

In paper [16] we restricted our attention to the sufficiently small λ s. We emphasized that Eq. (8) may be further reparametrized in terms of the time which would measure a recovery from the QC singularity,

$$\lambda \rightarrow \lambda_n(t) = t + t^2 + \dots + t^{J-1} + G_n t^J \quad n = 1, 2, \dots, N, \quad J = [N/2]. \quad (9)$$

Formally, the time-parameter $t \in (0, \infty)$ was redundant but its presence enabled us to treat Hamiltonian $H_{(PT)}^{(N)}$ as manifestly time-dependent.

3.2 Metric Θ near the singularity

In the above picture the evolution is interpreted as the motion of the system away from QC, towards a more stable dynamical regime. Parametrization (9) proves useful, first of all, at the very short times $t \ll 1$ at which it effectively rescales and magnifies the interior of \mathcal{D} in the EEP vicinity. Simultaneously, such an *ad hoc* change of scale did not lower the number of degrees of freedom – one could still work with as many as J coupling constants $G_n \geq 0$.

With the trivial selection of the new coupling constants $G_n = 0$ the presence of the time t allowed one for a one-parametric change of the dynamics. After the hypothetical start of the evolution at the QC singularity $t = 0$ one reaches the manifestly Hermitian harmonic-oscillator limit at $\lambda(t) = 1$. In this sense parameter $\lambda \in (0, 1)$ may be perceived as an alternative, simplified measure of the QC-recovery time.

At any fixed value of N and for any given Hamiltonian the physical contents of the theory varies with the inner product (1). The reconstruction of all of the eligible metrics Θ may be found summarized in our dedicated work [22]. It is based on a replacement of

matrix $H_{(PT)}^{(N)}$ by its transposition in Schrödinger equation,

$$\left[H_{(PT)}^{(N)} \right]^\dagger |\psi_n^{(N)}\rangle\rangle = E_n^{(N)} |\psi_n^{(N)}\rangle\rangle, \quad n = 0, 1, \dots, N-1. \quad (10)$$

The complete solution of this equation opens the way towards the reconstruction of any metric from its spectral representation

$$\Theta_{(\vec{\kappa})}^{(N)}(t) = \sum_{n=1}^N |\psi_n^{(N)}(t)\rangle\rangle \kappa_n \langle\langle \psi_n^{(N)}(t) |. \quad (11)$$

All of the parameters $\kappa_n > 0$ are freely variable. This, as we already mentioned, is an ambiguity which is usually being suppressed via a more detailed information about the system (cf., e.g., Ref. [3] for explanation).

3.3 The ambiguity of the metric

Let us pick up $N = 2$ and study the possible consequences of the ambiguity of metric (11) in more detail. Firstly, let us change the variables, $t \rightarrow r = r(t) = \sqrt{t} > 0$, yielding

$$\left[H_{(PT)}^{(N)} \right]^\dagger = \begin{bmatrix} -1 & -\sqrt{1-r^2} \\ \sqrt{1-r^2} & 1 \end{bmatrix}. \quad (12)$$

In terms of the pair of abbreviations $u = \sqrt{1-r}$ and $v = \sqrt{1+r}$ we may then follow paper I and calculate the maximal and minimal eigenvalues $E_+ = r$ and $E_- = -r$ of (12) as well as the related respective eigenvectors

$$|\psi_+\rangle\rangle = [\sqrt{1-r}, -\sqrt{1+r}] = [u, -v], \quad (13)$$

$$|\psi_-\rangle\rangle = [\sqrt{1+r}, -\sqrt{1-r}] = [v, -u]. \quad (14)$$

It remains for us to insert vectors (13) and (14) in the spectral expansion of the metric. Once we fix an inessential overall factor and denote $\kappa_+ = \sin \alpha$ and $\kappa_- = \cos \alpha$ with $0 < \alpha < \pi/2$ we get the general $N = 2$ metric-operator matrix of paper [23],

$$\Theta = \Theta_{[\alpha]}^{(2)}(r^2) = \begin{bmatrix} 1 + r \cos 2\alpha & -\sqrt{1-r^2} \\ -\sqrt{1-r^2} & 1 - r \cos 2\alpha \end{bmatrix}. \quad (15)$$

Its elementary form facilitates the direct determination of its eigenvalues,

$$\theta_\pm = 1 \pm \sqrt{1 - r^2 \sin^2 2\alpha}. \quad (16)$$

One easily verifies that the requirement of the necessary positivity of these eigenvalues is trivially satisfied at any square-root-time $r = \sqrt{t}$ such that $0 < r < 1$.

We may conclude that the standard probabilistic interpretation of our time-dependent $N = 2$ THS QC quantum model is determined not only by its one-parametric Hamiltonian (12) but also by the specification of the concrete value of variable α . Via Eq. (15) this choice selects one of the eligible inner products (1). This makes the Hilbert space of states fully defined and unique.

4 Minimal anisotropy constraint

The concept of quantum catastrophes perceived as infinitesimally short processes is certainly incomplete. Indeed, we saw that the full picture of evolution contains multiple Hamiltonian-independent parameters, sampled by α of Eq. (15). Thus, the theory based on the mere knowledge of the Hamiltonian would still admit many phenomenologically nonequivalent predictions. At non-infinitesimal times t the evolution would remain undetermined. Naturally, such a theory would be unsatisfactory. The ambiguity must be removed via additional, physics-based constraints [3].

One of the most popular options may be found advocated, e.g., in review [4]. It makes use of an additional postulate of the observability of a charge. In what follows, we intend to propose another, conceptually different form of the necessary constraint. We shall be able to make the metric unique via a new, QC-related postulate.

4.1 A reversal of the arrow of time

In paper I we were only able to construct the unique and closed-form metrics up to $N = 3$ or, in the implicit spectral-expansion form (11), up to $N = 5$. The puzzling arbitrariness of the choice of parameters $\vec{\kappa}$ in Eq. (11) was successfully eliminated by means of an *ad hoc* requirement of a maximal simplicity of the formulae for the matrix elements of the metrics. The main motivation of such a rather formal method of suppression of the unwanted ambiguity of the physical predictions was rather vague. The main argument supporting this strategy was that it perceivably facilitated the extrapolations of the formulae with respect to the dimension N .

The entirely artificial nature of such a half-intuitive and model-dependent choice of $\vec{\kappa}$ s

appeared to be one of the strongest motivations of a renewal of our analysis of the problem of ambiguity. Our efforts climaxed in the successful resolution of the puzzle which will be described in what follows. Our new recipe will be based, first of all, on a removal of the restriction of attention just to a small vicinity of the QC instant $t = 0$. We shall extend the applicability of our toy models to the explicit description of the whole QC-generating process.

The change of philosophy will force us to replace the not too suitable time-of-recovery t (or, if you wish, the time-like parameter $\lambda(t)$ of Eq. (9)) by the new time variable τ . It will run in opposite direction, i.e., from the $\tau = 0$ instant at which our Hamiltonian was still entirely standard (i.e., in our toy models, diagonal) to the QC $\tau = 1$ instant at which, by construction, our Hamiltonians become non-acceptable – fully non-diagonalizable and merely Jordan-block representable [24].

4.2 Metrics as functions of τ

The choice of the concrete definition of the new “time of collapse” $\tau = \sqrt{1 - \lambda}$ will enable us to treat the fall of our system into its singularity as a process which started at $\tau = 0$, i.e., long time before the catastrophe. A disadvantage of the replacement of t by τ might be seen in the fact that the construction of the metric appears much easier at the small QC-distances λ . Indeed, in the QC limit $\tau \rightarrow 1$ the time-dependent vectors $|\psi_n\rangle$ are getting mutually parallel so that the operator (11) itself degenerates to a particularly elementary matrix of rank one. Fortunately, near the opposite extreme of $\tau = 0$ the transition $t \rightarrow \tau$ simplifies the Hamiltonians,

$$\begin{aligned}
 H^{(2)}(\tau) &= \begin{bmatrix} -1 & \tau \\ -\tau & 1 \end{bmatrix}, & H^{(3)}(\tau) &= \begin{bmatrix} -2 & \sqrt{2}\tau & 0 \\ -\sqrt{2}\tau & 0 & \sqrt{2}\tau \\ 0 & -\sqrt{2}\tau & 2 \end{bmatrix}, \\
 H^{(4)}(\tau) &= \begin{bmatrix} -3 & \sqrt{3}\tau & 0 & 0 \\ -\sqrt{3}\tau & -1 & 2\tau & 0 \\ 0 & -2\tau & 1 & \sqrt{3}\tau \\ 0 & 0 & -\sqrt{3}\tau & 3 \end{bmatrix}, \dots \dots
 \end{aligned} \tag{17}$$

In other words, one feels tempted to replace the explicit use of the spectral expansion (11) by some less subtle, brute-force construction techniques which make the use of the tridiagonal-matrix form of the Hamiltonians [25]. Such an idea proved truly productive.

Theorem 1 *The metrics $\Theta^{(N)}(\tau)$ compatible with the respective Hamiltonians (17) may be sought in the generic form*

$$\Theta^{(N)}(\tau) = \sum_{j=1}^N (-\tau)^{j-1} \mathcal{M}^{(N)}(j) \quad (18)$$

containing some suitable sparse-matrix coefficients

$$\mathcal{M}^{(N)}(1) = \begin{bmatrix} \alpha_{11}(1) & 0 & \dots & 0 \\ 0 & \alpha_{12}(1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_{1N}(1) \end{bmatrix}, \quad (19)$$

$$\mathcal{M}^{(N)}(2) = \begin{bmatrix} 0 & \alpha_{11}(2) & 0 & \dots & \dots & 0 \\ \alpha_{21}(2) & 0 & \alpha_{12}(2) & 0 & \dots & 0 \\ 0 & \alpha_{22}(2) & 0 & \alpha_{13}(2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_{2,N-2}(2) & 0 & \alpha_{1,N-1}(2) \\ 0 & \dots & \dots & 0 & \alpha_{2,N-1}(2) & 0 \end{bmatrix}, \quad (20)$$

$$\mathcal{M}^{(N)}(3) = \begin{bmatrix} 0 & 0 & \alpha_{11}(3) & 0 & \dots & \dots & 0 \\ 0 & \alpha_{21}(3) & 0 & \alpha_{12}(3) & 0 & \dots & 0 \\ \alpha_{31}(3) & 0 & \alpha_{22}(3) & 0 & \alpha_{13}(3) & \ddots & \vdots \\ 0 & \alpha_{32}(3) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & \alpha_{2,N-3}(3) & 0 & \alpha_{1,N-2}(3) \\ \vdots & \dots & 0 & \alpha_{3,N-3}(3) & 0 & \alpha_{2,N-2}(3) & 0 \\ 0 & \dots & \dots & 0 & \alpha_{3,N-2}(3) & 0 & 0 \end{bmatrix}, \quad (21)$$

etc. In this way, the set of all of the non-vanishing elements of matrix $\mathcal{M}^{(N)}(k)$ may be arranged into an auxiliary, k by $(N - k + 1)$ –dimensional array

$$\alpha(k) = \begin{bmatrix} \alpha_{11}(k) & \alpha_{12}(k) & \alpha_{13}(k) & \dots & \alpha_{1,N-k+1}(k) \\ \alpha_{21}(k) & \alpha_{22}(k) & \alpha_{23}(k) & \dots & \alpha_{2,N-k+1}(k) \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{k1}(k) & \alpha_{k2}(k) & \alpha_{k3}(k) & \dots & \alpha_{k,N-k+1}(k) \end{bmatrix}. \quad (22)$$

at any $k = 1, 2, \dots, N$, with $\alpha_{11}(k) = \mathcal{M}_{1k}^{(N)}(k)$, etc.

Proof follows from the inspection of the set of the N^2 Dieudonné's linear algebraic compatibility relations written in the matrix form

$$H^\dagger \Theta = \Theta H \quad (23)$$

(not all of which are independent - cf. Ref. [22] for details). \square

4.3 The physics of the onset of degeneracy

From the point of view of potential applications of formula (18) it is important that at the beginning of the fall into instability (i.e., at the far-from-QC instant $\tau = 0$) the Hamiltonians (17) will all coincide with the respective truncated and diagonal (i.e., Hermitian) harmonic-oscillator-like matrices. In this picture the spectrum of energies $E_n^{(N)}(\tau)$ will remain real but shrinking with the growth of the innovated time τ . In the limit $\tau \rightarrow 1$, i.e., at the very end of the fall of the system into QC singularity, the spectrum becomes completely degenerate, $E_n^{(N)}(1) = 0$, $n = 0, 1, \dots, N - 1$.

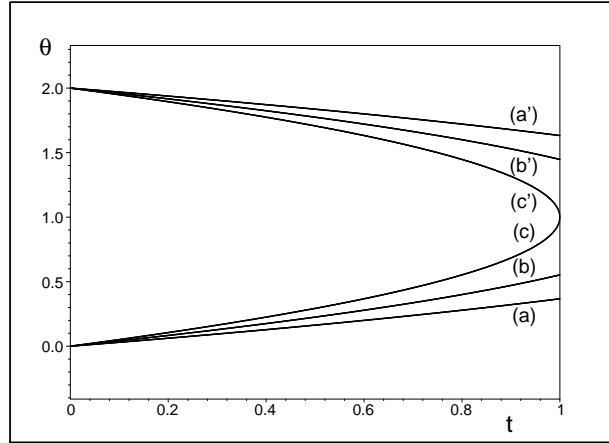


Figure 1: Eigenvalues (16) of the $N = 2$ metric (15) for the recovery time $t = r^2 \in (0, 1)$ and for the parameters (a) $\sin^2 2\alpha = 3/5$, (b) $\sin^2 2\alpha = 4/5$ and (c) $\sin^2 2\alpha = 1$. The desirable disappearance of the anisotropy at the full-recovery time $t = 1$ only occurs for the minimal-anisotropy choice of $\sin^2 2\alpha = 1$.

At the latter instant the Hamiltonian $H^{(N)}(1)$ ceases to be diagonalizable and loses its standard physical tractability and interpretation. *Vice versa*, from the point of view of physics the description of the evolution as generated by $H^{(N)}(\tau)$ will change at $\tau = 1$, requiring an introduction of some new degrees of freedom beyond this instant, i.e., at

$\tau > 1$. The study of such a switch to a new form of Hamiltonian at later times lies already beyond the scope of the present paper. Examples of such a scenario may be found, e.g., in magnetohydrodynamics [26].

Temporarily, let us now return, in the context of interpretations, to the times of recovery t or $\lambda(t)$. In these variables, the motion beyond the end-of-the-interval $\lambda(t) = 1$ appears much less exotic. Typically, the energies would not feel the change at all (cf., e.g., Eqs. (5) – (7)). Still, the values of $\lambda = \lambda^{(outer)} > 1$ remain *mathematically* exceptional because in the “outer” interval of parameters our Hamiltonian matrices change their form thoroughly: they become complex and Hermitian.

We are now coming very close to the key message of our present paper, referring to the fact that in the “outer” interval of parameters our Hamiltonians become manifestly Hermitian in usual sense. For this reason it seems most natural to require that the metric remains trivial in such an interval, $\Theta^{(outer)} \equiv I$. In other words, we *suggest* to match the separate non-Hermitian and Hermitian dynamical regimes strictly at the natural matching point with $\lambda(t) = 1$.

One of the consequences of such an innovative model-building decision may be seen in the subsequent most natural reinterpretation of the point of matching $\lambda(t) = 1$ as the time of onset of the whole QC scenario using $\lambda(t) \in (0, 1)$. The optimal QC-related metrics should be then required *continuous* at the instant $\lambda(t) = 1$, i.e., at the very onset of the whole process of the QC degeneracy,

$$\lim_{\tau \rightarrow 0} \Theta^{(N)}(\tau) = I. \quad (24)$$

In a climax of this paragraph we must add and emphasize, incidentally, that the new restriction (24) proves strictly satisfied by the family of the formally admissible metrics in their unique extrapolation-friendly form as introduced in paper I.

The latter observation may be most immediately illustrated via the $N = 2$ model again. Recalling the general set of the metrics (15) which appear numbered by the single kinematical real variable α , we may conclude that each deviation of the latter free parameter from its extrapolation friendly value of paper I and from its anisotropy minimizing value which follows from formula (16) also, simultaneously, violates the present onset-continuity constraint (24). In this sense, all of the three forms of the constraint may be perceived equivalent. For general dimension N , their most immediate test will be provided by Eq. (24) of course.

Figure 1 offers the simplest $N = 2$ illustration of the fine-tuning nature of the coincidence. We see there that the influence of α changes from very weak in the QC vicinity (i.e., at $t \ll 1$) to very strong near the Hermitian LHO dynamical regime (where $\lambda(t) = 1$). In such a graphical representation we also see that the extrapolation-friendly choice of $\alpha = \pi/4$ of paper I is truly exceptional. *Solely* for this choice of the free parameter the difference between the eigenvalues of the metric will strictly vanish at $\lambda(t) = 1$. We may just repeat that for the value of $\alpha = \pi/4$ the $\tau = 0$ instant really carries the meaning of a Hermitian onset of the fall into QC singularity.

Once we preserve the latter exceptional choice of the parameter $\alpha = \pi/4$ at all times $t \in (0, 1)$, the difference between the two eigenvalues of the $N = 2$ metric (measuring a Hilbert-space anisotropy) remains always minimized (cf. both Fig. 1 and Eq. (16)). In this formulation, the selection of a “minimal Hilbert-space anisotropy” principle certainly becomes much less model-dependent. Moreover, the uniqueness of the $N = 2$ choice of $\alpha = \pi/4$ may be, *mutatis mutandis*, extended to all of the higher matrix dimensions N . After all, one might most easily proceed from the point of the onset $\tau = 0$, i.e., from the very end $\lambda = 1$ of the recovery of the system from the QC dynamical regime, by connecting our specific removal of the mathematical ambiguity of the metric directly with the well founded physical trivalization (24) of the metric in the Hermitian-Hamiltonian limit.

5 Hilbert spaces with minimal anisotropy

In paper [22] we thoroughly discussed the general recipe by which suitable metrics Θ may always be assigned to a given Hamiltonian H via Eq. (11). We reemphasized there that such a simple-minded construction is always ambiguous [3] but that for the finite space-dimensions $N < \infty$ the key merit of the recipe lies in the sufficiency of the solution of the auxiliary conjugate Schrödinger equation (10).

In paper I we applied this philosophy to Hamiltonians (17). Up to $N = 5$, we managed to evaluate the necessary input ketket-eigenvectors $|\psi_n^{(N)}\rangle\rangle$. We also revealed a very regular sparse-matrix pattern in these formulae [cf. Eq. Nr. (10) in paper I]. Finally, we emphasized the universality of the recipe (11) yielding all the admissible metrics. At the same time, our interest remained restricted to the mere vicinity of the QC dynamical regime.

In our present paper we shifted the emphasis from the universality of spectral for-

mula (11) to the alternative possibility of its explicit resummation. Let us now test this strategy at $N = 2$ once more.

5.1 $N = 2$, revisited

The explicit and universal construction of the metric $\Theta^{(N)}(\tau)$ as presented in paper I (i.e., the one performed via the auxiliary Schrödinger Eq. (10)) is not too easy even at $N = 2$, i.e., for our first nontrivial QC-related Hamiltonian matrix

$$H^{(2)}(\tau) = \begin{bmatrix} -1 & \tau \\ -\tau & 1 \end{bmatrix}. \quad (25)$$

This may make the efficiency of this construction comparable with the direct solution of Eq. (23). For a verification, let us now return to the latter methodical possibility and let us try to rederive, say, the complete family of the $N = 2$ metrics $\Theta^{(2)}$ as already known from Ref. [23] and also from subsection 3.3 above.

In the real-matrix ansatz with the subscripted vector $\vec{\kappa}$ containing two arbitrary positive components,

$$\Theta_{(\vec{\kappa})}^{(2)}(\tau) = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad (26)$$

we shall, first of all, fix an overall multiplication constant by setting the determinant equal to one. This enables us to put $b = \sinh \nu$ and choose $\varepsilon = \pm 1$ in $a = \varepsilon \cosh \nu \exp \varrho$ and $d = \varepsilon \cosh \nu \exp(-\varrho)$ (both the new parameters ν and ϱ are assumed real). As long as the metric must be positive we may only use $\varepsilon = 1$. Finally we check that the matrix constraint (23) degenerates to the single, time-reparametrization item

$$\tau = -\frac{\tanh \nu}{\cosh \varrho}. \quad (27)$$

Our conclusion is that for any given $\tau \in (0, 1)$ we may choose *any* real $\varrho \in (0, \varrho_{max})$ (note that this is the parameter which makes the main diagonal of the metric asymmetric). This choice enables us to evaluate $\nu = \nu(\tau, \varrho)$ from the latter equation (this implies that at a fixed time, the value of ϱ_{max} must be such that $\cosh \varrho_{max} = 1/\tau$). Summarizing, we may set $\alpha_{11}(1) = \cosh \nu \exp \varrho$, $\alpha_{12}(1) = \cosh \nu \exp(-\varrho)$ and $\alpha_{11}(2) = \sinh \nu$ in Eq. (18) at $N = 2$. The resulting eigenvalues of the metric are both, by construction, positive,

$$\theta_{\pm} = \cosh \nu \cosh \varrho \pm \sqrt{\cosh^2 \nu \cosh^2 \varrho - 1}. \quad (28)$$

At the very start of the fall of the system into the catastrophe, i.e., at $\tau = 0$ one has $\varrho_{max}(0) = \infty$ so that there is no upper bound imposed upon $\varrho(0)$. Still, as long as one might like to have the trivial, isotropic initial value of $\Theta^{(2)}(0) \sim I$ (implying the special choice of $\nu(0) = 0$ and $\varrho(0) = 0$) the resulting metric becomes, up to the above-mentioned irrelevant overall multiplication factor, unique at $\tau = 0$.

During the subsequent growth of $\tau < 1$ the requirement of the minimization of the anisotropy leads to the rule $\varrho(\tau) = 0$ (cf. Eq. (28)) so that the remaining variable $\nu < 0$ may now be interpreted as another (viz., rescaled and, incidentally, inverted, cf. Eq. (27)) version of the time of the QC degeneracy.

Once we return to the standard variables we get our unique and minimally anisotropic metric in the virtually trivial form as given in paper I,

$$\Theta^{(2)} = \begin{bmatrix} 1 & -\tau \\ -\tau & 1 \end{bmatrix} = I - \tau J.$$

From this formula we may deduce the special, minimally anisotropic version of eigenvalues in the form compatible with their “more anisotropic” generalization (16).

5.2 $N = 3$

Whenever one tries to move to the higher matrix dimensions N one encounters the technical problem of the increasing multitude of various parameters. Thus, in the first nontrivial case with $N = 3$ let us first follow the $N = 2$ guidance (cf. the ultimate choice of $\varrho = 0$ in the preceding paragraph 5.1) and let us omit the discussion of the metrics with an asymmetric form of their main diagonal.

Once we also keep ignoring the other, irrelevant though still existing overall factor, we are, after some straightforward manipulations using Eq. (23), left with the last free parameter g in the metric

$$\Theta^{(3)}(\tau) = \begin{bmatrix} 1 & -\sqrt{2}g\tau & g\tau^2 \\ -\sqrt{2}g\tau & 2g - 1 + g\tau^2 & -\sqrt{2}g\tau \\ g\tau^2 & -\sqrt{2}g\tau & 1 \end{bmatrix}. \quad (29)$$

Among its three readily obtainable eigenvalues

$$\theta_1 = g\tau^2 + g - \sqrt{4g^2\tau^2 + g^2 - 2g + 1}, \quad \theta_2 = 1 - g\tau^2, \quad \theta_3 = g\tau^2 + g + \sqrt{4g^2\tau^2 + g^2 - 2g + 1} \quad (30)$$

the middle one (i. e., the inverted parabola in τ) remains positive for the parameters $g < 1/\tau^2$ while the change of the sign of the remaining pair may take place at the curves $g = 1/\tau^2$ and $g = 1/(2 - \tau^2)$ in the $g - \tau$ plane. Figures 2 - 4 may be recalled as showing why the correct and unique choice of the parameter is $g = 1$, yielding again the unique metric of paper I,

$$\Theta^{(3)} = I - \tau \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} + \tau^2 J \quad (31)$$

with the correct and expected τ -dependence of the eigenvalues as given by Eq. (30).

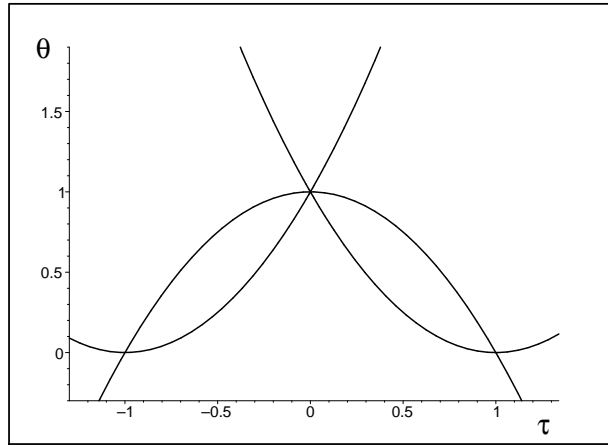


Figure 2: Eigenvalues (30) of the $N = 3$ metric (29) in an extended range of time of fall τ and for the “correct” parameter $g = 1$.

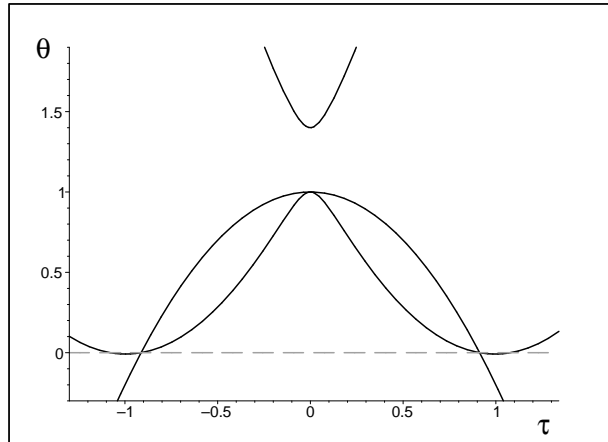


Figure 3: Eigenvalues (30) of the $N = 3$ metric (29) in an extended range of time of fall τ and for a “too large” parameter $g = 1.2$. Notice that the metric is not isotropic even at the onset time $\tau = 0$.

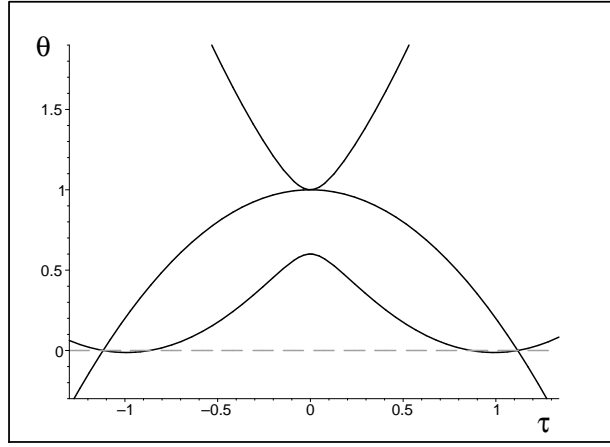


Figure 4: Eigenvalues (30) of the $N = 3$ metric (29) in an extended range of time of fall τ and for a “too small” parameter $g = 0.8$. Notice that in this regime the metric ceases to be acceptable (i.e., positive) *long before* the QC time $\tau = 1$ is reached.

5.3 $N = 4$

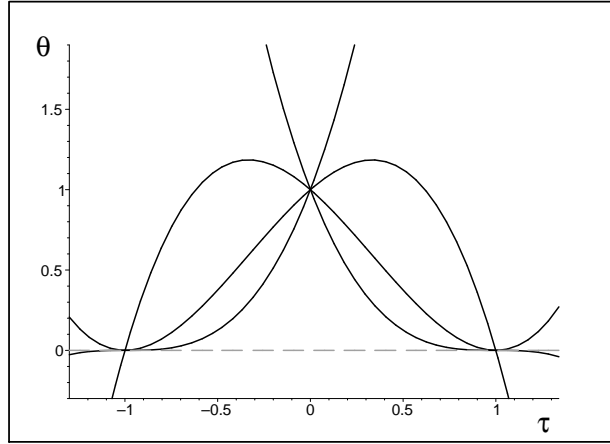


Figure 5: Eigenvalues (33) of the “correct” $N = 4$ metric (32) in an extended range of time of fall τ .

In the next step of our constructive considerations we are getting beyond the formulae derived in paper I. Fortunately, already the $N = 3$ formula (31) itself offers a clear hint of an extrapolation so that it just proves sufficient to verify that the next metric

$$\Theta^{(4)} = \begin{bmatrix} 1 & -\sqrt{3}\tau & \sqrt{3}\tau^2 & -\tau^3 \\ -\sqrt{3}\tau & 1 + 2\tau^2 & -2\tau - \tau^3 & \sqrt{3}\tau^2 \\ \sqrt{3}\tau^2 & -2\tau - \tau^3 & 1 + 2\tau^2 & -\sqrt{3}\tau \\ -\tau^3 & \sqrt{3}\tau^2 & -\sqrt{3}\tau & 1 \end{bmatrix} \quad (32)$$

obeys all the necessary and sufficient requirements including, first of all, the Dieudonné

Eq. (23). Once we also deduce the τ -dependent eigenvalues of this candidate for the metric at $N = 4$ we may immediately see that they behave as they should,

$$\{\theta_1, \dots, \theta_4\} = \{1 - 3\tau + 3\tau^2 - \tau^3, 1 - \tau - \tau^2 + \tau^3, 1 + 3\tau + 3\tau^2 + \tau^3, 1 + \tau - \tau^2 - \tau^3\} \quad (33)$$

(cf. also Fig. 5 for a graphical illustration of their correct QC behaviour at $\tau = 1$).

6 Extrapolations in N

6.1 Metrics between $N = 5$ and $N = 7$

We may now combine the results of preceding section with the contents of Theorem 1. Using an elementary insertion in Eq. (23) we may easily prove that in the expansions (18) of the metrics with minimal anisotropy the diagonal-matrix coefficients (19) may be defined, at all N , by the elementary formula

$$\alpha_{1n}(1) = 1, \quad n = 1, 2, \dots, N.$$

Similarly, the closed formula is also available for the antidiagonal coefficients in $\mathcal{M}^{(N)}(N)$,

$$\alpha_{n1}(N) = 1, \quad n = 1, 2, \dots, N.$$

Next, the bidiagonal matrix coefficients (20) may be defined, at all N , by the slightly less elementary general formula

$$\alpha_{1n}(2) = \alpha_{2n}(2) = \sqrt{n(N-n)}, \quad n = 1, 2, \dots, N-1.$$

Due to the easily verified symmetry, the analogous formula exists for the coefficients in $\mathcal{M}^{(N)}(N-1)$,

$$\alpha_{n1}(N-1) = \alpha_{n2}(N-1) = \sqrt{n(N-n)}, \quad n = 1, 2, \dots, N-1.$$

Up to now, unfortunately, we did not succeed in an extension of these observations to the tridiagonal sparse matrix coefficients (21), etc. Nevertheless, we believe that the task is not impossible. This belief seems supported by Theorem 1, i.e., by the reducibility of the N by N matrices $\mathcal{M}^{(N)}(k)$ with $k = 3, 4, \dots$ to the respective auxiliary k by $N - k + 1$ arrays containing the non-vanishing matrix elements $\alpha_{jm}(k)$ of $\mathcal{M}^{(N)}(k)$.

The first missing set of coefficients occurs at $N = 5$. Its values

$$\alpha_{11}(3) = \alpha_{13}(3) = \alpha_{31}(3) = \alpha_{33}(3) = \sqrt{6},$$

$$\alpha_{12}(3) = \alpha_{21}(3) = \alpha_{23}(3) = \alpha_{32}(3) = 3, \quad \alpha_{22}(3) = 4.$$

may still be found evaluated, albeit in different context, in paper I. Naturally, this definition should be better rewritten in the much more compact form of the array

$$\alpha(3) = \begin{bmatrix} \sqrt{6} & 3 & \sqrt{6} \\ 3 & 4 & 3 \\ \sqrt{6} & 3 & \sqrt{6} \end{bmatrix}, \quad N = 5. \quad (34)$$

It makes sense to complemented this result by the next, $N = 6$ formula

$$\alpha_{11}(3) = \alpha_{14}(3) = \alpha_{31}(3) = \alpha_{34}(3) = \sqrt{10}, \quad \alpha_{21}(3) = \alpha_{24}(3) = 4,$$

$$\alpha_{12}(3) = \alpha_{13}(3) = \alpha_{32}(3) = \alpha_{33}(3) = 3\sqrt{2}, \quad \alpha_{22}(3) = \alpha_{23}(3) = 6$$

which we derived using the brute force construction based on Eq. (23). It again deserves the compact presentation as the array

$$\alpha(3) = \begin{bmatrix} \sqrt{10} & 3\sqrt{2} & 3\sqrt{2} & \sqrt{10} \\ 4 & 6 & 6 & 4 \\ \sqrt{10} & 3\sqrt{2} & 3\sqrt{2} & \sqrt{10} \end{bmatrix}, \quad N = 6. \quad (35)$$

The closed form of the latter result indicates that there might exist a not too complicated extrapolation recipe, with the help of which we would be able to determine the unique, minimally anisotropic metric at any dimension N . This belief seems further supported by the regularity and apparent extrapolation-friendliness of the next two sparse-matrix “missing” coefficients

$$\mathcal{M}^{(7)}(3) = \begin{bmatrix} 0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & \sqrt{30} & 0 & 0 & 0 \\ \sqrt{15} & 0 & 8 & 0 & 6 & 0 & 0 \\ 0 & \sqrt{30} & 0 & 9 & 0 & \sqrt{30} & 0 \\ 0 & 0 & 6 & 0 & 8 & 0 & \sqrt{15} \\ 0 & 0 & 0 & \sqrt{30} & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{M}^{(7)}(4) = \begin{bmatrix} 0 & 0 & 0 & 2\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{10} & 0 & 2\sqrt{10} & 0 & 0 \\ 0 & 2\sqrt{10} & 0 & 6\sqrt{3} & 0 & 2\sqrt{10} & 0 \\ 2\sqrt{5} & 0 & 6\sqrt{3} & 0 & 6\sqrt{3} & 0 & 2\sqrt{5} \\ 0 & 2\sqrt{10} & 0 & 6\sqrt{3} & 0 & 2\sqrt{10} & 0 \\ 0 & 0 & 2\sqrt{10} & 0 & 2\sqrt{10} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{5} & 0 & 0 & 0 \end{bmatrix}.$$

Again, they were obtained, with the assistance of the computerized symbolic manipulations, by the brute-force solution of the set of 49 linear algebraic Eqs. (23).

6.2 Eigenvalues of the metrics

Table 1: Pascal triangle for coefficients $C_{1n}^{(N)}$ in Eq. (36)

N								
1	1							
2	1		1					
3	1			2	1			
4	1				3	3	1	
5	1					4	6	4
6	1						5	10
7	1							6
8	1	7	21	35	35	21	7	1
⋮	...							

In the above-described constructions of the N by N matrices of metric $\Theta^{(N)}$ we did not manage to find, unfortunately, any obvious general extrapolation tendency or pattern. For this reason, we turned our attention from the matrices to the perceivably simpler-to-display N -plets of their eigenvalues $\theta_n^{(N)}(\tau)$.

Table 2: Pascal-like triangle for coefficients $C_{2n}^{(N)}$ in Eq. (36)

N								
2				1	-1			
3				1	0	-1		
4			1	1	-1	-1		
5			1	2	0	-2	-1	
6		1	3	2	-2	-3	-1	
7	1	4	5	0	-5	-4	-1	
8	1	5	9	5	-5	-9	-5	-1
\vdots					...			

Proposition 2 *The time-dependent eigenvalues of $\Theta^{(N)}$ may be written in the form*

$$\theta_n^{(N)} = \theta_n^{(N)}(\tau) = \sum_{k=1}^N C_{nk}^{(N)} \tau^{k-1} \quad (36)$$

where, up to $N = 8$, the numerically evaluated values of the coefficients $C_{nk}^{(N)}$ may be found summarized in the Pascal-like tables 1 - 4.

In the final step of our considerations let us now show that the closed-form eigenvalues of the metric can be obtained at any matrix dimension N .

Theorem 3 *The time-dependent eigenvalues of $\Theta^{(N)}$ are given by formula*

$$\theta_k^{(N)}(\tau) = \sum_{m=1}^N C_{km}^{(N)} \tau^{m-1}, \quad k = 1, 2, \dots, N$$

where $C_{1n}^{(N)} = \binom{N-1}{n-1}$, $C_{2n}^{(N)} = \binom{N-2}{n-1} - \binom{N-2}{n-2}$ and, in general,

$$C_{kn}^{(N)} = \sum_{p=1}^k (-1)^{p-1} \binom{k-1}{p-1} \binom{N-k}{n-p}, \quad k, n = 1, 2, \dots, N.$$

Proof is straightforward and proceeds by mathematical induction. □

Table 3: Pascal-like triangle for coefficients $C_{3n}^{(N)}$ in Eq. (36)

N								
3				1	-2	1		
4				1	-1	-1	1	
5			1	0	-2	0	1	
6		1	1	-2	-2	1	1	
7		1	2	-1	-4	-1	2	1
8	1	3	1	-5	-5	1	3	1
\vdots						...		

Table 4: Pascal-like triangle for coefficients $C_{4n}^{(N)}$ in Eq. (36)

N								
4				1	-3	3	-1	
5			1	-2	0	2	-1	
6		1	-1	-2	2	1	-1	
7		1	0	-3	0	3	0	-1
8	1	1	-3	-3	3	3	-1	-1
\vdots						...		

7 Discussion

We may summarize that our present, toy-model-based generic scenario of the evolution of a quantum system \mathcal{S} sets the scene between its innocent-looking and safely Hermitian equidistant-energy-level onset prepared at the initial time-of-fall $\tau = 0$ and a full realization of the complete, N -tuple degeneracy of the energy levels at the final QC time $\tau = 1$.

According to the general principles of quantum theory in its THS form the states of the system \mathcal{S} are assumed described by a wave function ψ which evolves in time in a way which has been thoroughly described in Ref. [6]. In our present paper we omitted the repetition of the related discussion and never mentioned this purely technical and routine aspect of the model. We rather restricted our attention to the details of the interplay between

the time-dependence of the pre-selected benchmark Hamiltonians $H^{(N)}$ and of the exactly constructed and minimally anisotropic unique metric operators $\Theta^{(N)}$.

The set of our conclusions offered a compact and consistent picture of the process in which the not quite expected exact solvability of our toy model of paper I enabled us to extend its applicability to all times $\tau \in (0, 1)$ measuring the whole QC-preceding history which started, at $\tau = 0$, from an entirely standard, Hermitian-matrix equidistant-level initial Hamiltonian. Via our N -numbered family of benchmark models given and defined by the matrix pairs

$$(H^{(N)}, \Theta^{(N)})$$

we were able to explain the quantum-evolution fall in the benchmark, N -tuple level-degeneracy catastrophe as a consequence of the steady growth of the anisotropy of the underlying physical, manifestly time-dependent Hilbert space $\mathcal{H}^{(S)}$.

We found it natural to characterize the extent of the latter Hilbert-space anisotropy by the non-coincidence and certain “spread” of the set of the eigenvalues $\theta_n^{(N)}$ of the pre-selected physical metric $\Theta^{(N)}$. On this background we decided to make the metric unique by means of the requirement of the minimization of this spread at any time τ and, more explicitly, by the ultimate and decisive requirement of the zero limit of the special spread measure $\rho = \max(\theta_n^{(N)} - \theta_m^{(N)})$ at the onset $\tau = 0$ of the whole hypothetical evolution process. On this background we finally arrived at the family of benchmark QC models in which our exceptional and unique metric Θ remained nontrivial during all times $\tau \in (0, 1)$. At the beginning of the QC history with $\tau = 0$, this metric was assumed to be matched to the trivial and most common isotropic metric $\Theta^{(Dirac)} = I$. In such a manner we arrived at the benchmark quantum representation of the catastrophe in which the fall into the N -tuple degeneracy as running in the interval of time $\tau \in (0, 1)$ may be perceived as initiated and smoothly “switched on”, at $\tau = 0$, to its arbitrary hypothetical unitary-evolution $\tau < 0$ prehistory.

Acknowledgments

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